Exploiting Don't Cares to Minimize *BMDs

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Abstract

*We present for the first time methods to minimize *BMDs exploiting don't care conditions. These minimization methods can be used during the verification of circuits by *BMDs. By changing function values for input vectors, which are in the don't care set, smaller *BMDs can be computed to keep peak memory consumption during *BMD construction as low as possible. We determine the complexity of the problem of don't care minimization for *BMDs and thus justify the use of heuristics to approximate the solution. Preliminary experimental results prove our heuristcs to be very effective in minimizing *BMD sizes.*

1 Introduction

One of the most important tasks during the design of *Integrated Circuits* is the verification of an implemented circuit, i.e., the check whether the implementation fulfills its specification.

In the last few years several methods based on *Decision Diagrams* (DDs) have been proposed [16, 4, 15] to perform verification. The idea is to transform both implementation and specification of a combinational circuit into a DD. Then, due to the canonicity of the DD representation, the equivalence check for specification and implementation reduces to the check whether the corresponding DDs are identical.

The most popular data structure in this context were *Binary Decision Diagrams* (BDDs) [3]. They were applied successfully e.g. to the verification of control logic and integer adders. But there are functions of high practical relevance (e.g. integer multipliers), which cannot be represented efficiently by BDDs. To overcome the limitations of BDDs other types of DDs were defined, e.g. *Binary Moment Diagrams* (BMDs) and *Multiplicative* BMDs (*BMDs) [5], which are able to represent integer–valued pseudo Boolean functions $f : \{0,1\}^n \to \mathbb{Z}$ and which are especially suited for arithmetic functions.

When a circuit consists of several modules or subcircuits, existing methods to compute the *BMD representing the overall circuit compute *BMDs for the modules and combine these *BMDs to a *BMD for the overall circuit by *BMD operations [7]. Other methods use backward construction [10, 14] from the circuit outputs towards the inputs and compose step by step the *BMD for a gate of the current cut front into the *BMD for the intermediate result.

A potential, which has not been used in this process so far, is the knowledge that certain input combinations cannot be applied to subcircuits/modules. Input combinations, which cannot be applied to subcircuits, can be given as don't care informations in the circuit specification or can be computed as satisfiability don't cares by image computations [1]. These don't cares can be used to minimize *BMDs – either before combining the *BMDs for submodules by *BMD operations or in the backward construction method when the processing of a submodule, for which don't care informations are at hand, is finished. In this context the minimization of *BMDs by exploiting don't care informations aims at reducing the *BMD sizes to keep peak memory consumption as low as possible.

The problem we have to solve is to minimize a *BMD B for a function f_B under don't care conditions given by a characteristic function dc ($dc(x) = 1$, if x is a don't care vector, i.e. x cannot be applied to the subcircuit realizing f_B). Since dc is a Boolean function, we assume that it is represented by a BDD. Our task is to compute a *BMD B' realizing a function $f_{B'}$, such that $f_B(x) = f_{B'}(x)$ for all x with $dc(x) = 0$ and B' has a (nearly) minimum number of nodes among all *BMDs fulfilling this property.

To the best of our knowledge the heuristics presented in this paper are the first solution to this problem. For the minimization of BDDs under don't care conditions there is a number of methods in the literature, e.g. [9, 8, 6, 20, 19, 12]. However for *BMDs the problem seems to be more difficult, since due to the Davio decomposition in *BMDs a change of the function value for a single input vector (exploiting a don't care for this input vector) has not only a "local effect" in the Decision Diagram, but can affect larger parts of the *BMD (see Section 2). A paper which has some relations to our work in this sense is [22]. In that work FDDs [13] are minimized (which are also based on Davio decompositions). In fact our first method[1] to minimize *BMDs (which are representations of integer–valued functions) is somewhat similar to the minimization of FDDs in [22] (FDDs are representations of Boolean functions). Another related paper is [21], which minimizes Reed–Muller forms. However the method from [21], which decides, whether to flip the value for a subset of coefficients in the Reed–Muller spectrum from 0 to 1 (1 to 0) or not, with the goal to maximize the number of zeros in the Reed–Muller spectrum, is not applicable when the values are integers as for functions represented by *BMDs.

We developed two different methods for the minimization of *BMDs under don't care conditions. After Section 2, which gives some basic definitions and notations, we determine the complexity of the problem and present our two heuristic methods in Section 3. In Section 4 we give preliminary experimental results to evaluate the approaches. The minimization results are very promising. The first method was able to reduce *BMD sizes by 75% on the average, the second even by 79%. Finally, Section 5 concludes the paper and gives directions for future research.

2 Preliminaries

In this section we give a brief review of BDDs [3], BMDs and *BMDs [5]. BDDs are used to represent Boolean functions $f : \{0,1\}^n \to \{0,1\}$, and both BMDs and *BMDs represent integer–valued pseudo Boolean functions $f : \{0,1\}^n \to \mathbb{Z}$.

A BDD is a rooted directed acyclic graph $G = (V, E)$ with non empty node set V containing two types of nodes, *non-terminal* and *terminal* nodes. A non-terminal node v has as label a variable $index(v) \in \{x_1, \ldots, x_n\}$ and two children $low(v), high(v) \in V$. We call $low(v)$ also 0–*successor*(v) and $high(v)$ 1–*successor*(v). The edge leading to $low(v)$ ($high(v)$) is called low (high) edge of v. BDDs are ordered [3]. A terminal node v is labeled with a value $value(v) \in \{0,1\}$ and has no outgoing edges. The Boolean function $f_v : \{0,1\}^n \to \{0,1\}$ defined by a BDD node v is defined recursively: If v is a terminal node with $value(v) = c \in \{0,1\}$, then $f_v(x_1,\ldots,x_n) = c$ and if v is a non-terminal node with $index(v) = x_i$, then $f_v(x_1,\ldots,x_n) = \overline{x_i} \cdot f_{low(v)}(x_1,\ldots,\, x_n) + x_i \cdot f_{high(v)}(x_1,\ldots,x_n)$. (BDDs use the so-called Shannon decomposition.) The function represented by a BDD B is equal to the function represented by its root node v_{root}.

Like BDDs BMDs are based on a rooted directed acyclic graph. In contrast to BDDs the terminal nodes v are labeled with values $value(v) \in \mathbb{Z}$. The recursive definition of the pseudo Boolean function $f_v : \{0,1\}^n \to \mathbb{Z}$ represented by a BMD node v differs from BDDs: If v is a terminal node with $value(v) = c \in \mathbb{Z}$, then $f_v(x_1,\ldots,x_n) = c$ and if v is a non-terminal node with $index(v) = x_i$, then $f_v(x_1,\ldots,x_n) = f_{low(v)}(x_1,\ldots,x_n) + x_i \cdot f_{high(v)}(x_1,\ldots,x_n)$. BMDs use the so-called *positive Davio decomposition*. It follows from this recursive definition that the function represented by $low(v)$ is equal to $f_v|_{x_i=0}$[†], but in contrast to Shannon decomposition the function represented by $high(v)$ is

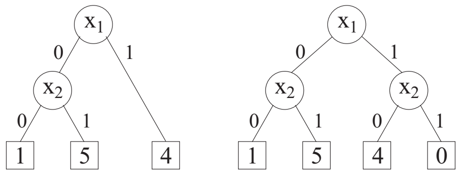
$$f_v|_{x_i=1} - f_v|_{x_i=0}. \tag{1}$$

Since BMDs use another decomposition type than BDDs (positive Davio decomposition instead of Shannon decomposition), the reduction rules to reduce the BMD sizes and to make BMDs a canonical data structure have to be changed compared to BDDs: As in the case of BDDs, if for terminal nodes v and $v' \in V$ $value(v) = value(v')$ or if for non-terminal nodes v and v' $index(v) = index(v')$, $low(v) = low(v')$ and $high(v) = high(v')$ then $v = v'$. However due to the Davio decomposition we have the reduction rule that in a reduced BMD there is no node $v \in V$ with $high(v) = t$, t terminal node with $value(t) = 0$.

For simplicity we assume in the following that the variables occur in the fixed order $x_1, \ldots, x_n$.

[1]see Section 3

[†]For a function, $f : \{0,1\}^n \to \mathbb{Z}$ $f_{x_i=0}$ ($f_{x_i=1}$) is the function which results from a substitution of x_i by constant 0 (1) and is called negative (positive) cofactor of f with respect to x_i.



(a) reduced (b) non-reduced

Figure 1: Example for a BMD.

To give a relation between nodes of a BMD B and cofactors of the function f_B represented by B, we define "the node which is reached by $(\epsilon_1, \ldots, \epsilon_l) \in \{0,1\}^l$ $(l \leq n)$":

To determine the node reached by $(\epsilon_1, \ldots, \epsilon_l)$ we start at the root node and follow the edges according to $(\epsilon_1, \ldots, \epsilon_l)$. If we are at a node v labeled with x_i and $\epsilon_i = 0$, then we follow the edge to $low(v)$ and if $\epsilon_i = 1$, we go to $high(v)$. Special attention has to be paid to the case, when ϵ_i*–successor*(v) has not label x_{i+1}. If in this case ϵ_i*–successor*(v) is a non–terminal, choose k with $x_k = index(\epsilon_i\text{–}successor(v))$ and if ϵ_i*–successor*(v) is a terminal choose $k = n+1$. Then we have to take into account, that in an non–reduced version of the BMD the edge leading to ϵ_i*–successor*(v) would be replaced by a path of nodes leading to ϵ_i*–successor*(v) where the labels are $x_{i+1}, \ldots, x_{k-1}$ and the high edges lead to the constant 0, respectively. Therefore we go to ϵ_i*–successor*(v) only if $\epsilon_{i+1} = \ldots = \epsilon_{k-1} = 0$, otherwise we say that the terminal 0 is reached by $(\epsilon_1, \ldots, \epsilon_l)$ (since 0 would be reached in a non–reduced version of the BMD). We call the node reached by $(\epsilon_1, \ldots, \epsilon_l)$ also $(\epsilon_1, \ldots, \epsilon_l)$–node and the function represented by this node $f_B^{(\epsilon_1, \ldots, \epsilon_l)}$.

Example 2.1 *Figure 1(a) shows an example of a* BMD *for function f with $f(0,0) = 1$, $f(0,1) = 6$, $f(1,0) = 5$ and $f(1,1) = 10$. The $(0,0)$–node is the terminal 1, the $(0,1)$–node is terminal 5, the $(1,0)$–node is terminal 4, but the $(1,1)$–node is terminal 0, since the high edge starting from the root leads to a terminal and not to a node with label x_2 and – as shown in Figure 1(b) – in the non–reduced* BMD *vector $(1,1)$ leads to terminal 0.*

Using (1) we can conclude the following lemma by induction:

Lemma 2.1 *Let B be a* BMD *representing a function $f_B : \{0,1\}^n \to \mathbb{Z}$ and let v be the $(\epsilon_1, \ldots, \epsilon_l)$–node $(l \leq n)$. Then the function $f_B^{(\epsilon_1, \ldots, \epsilon_l)}$ represented by v is equal to*

$$f_B^{(\epsilon_1, \ldots, \epsilon_l)} = \sum_{(\delta_1, \ldots, \delta_l) \leq (\epsilon_1, \ldots, \epsilon_l)} (-1)^{\sum_{i=1}^{l} (\epsilon_i - \delta_i)} f_B|_{x_1 = \delta_1, \ldots, x_l = \delta_l}. \tag{2}$$

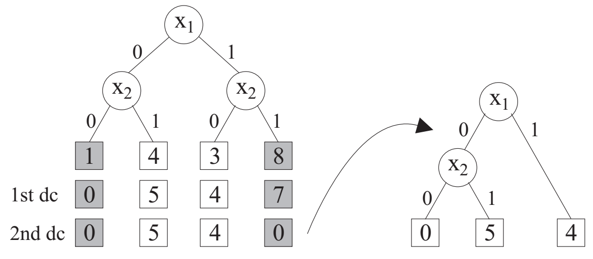
(For $\delta, \epsilon \in \{0,1\}^l : \delta \leq \epsilon$ iff $\delta_i \leq \epsilon_i$ $\forall 1 \leq i \leq l$.)

Lemma 2.1 shows that the change of the function f_B for a single input vector ϵ, i.e. the change of cofactor $f_B|_{x=\epsilon}$, has not only a "local effect" in the Decision Diagram, but affects all γ–nodes with $\epsilon \leq \gamma$.

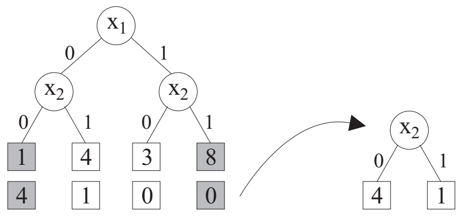
*BMDs were defined in [5] to further reduce the size of BMDs by increasing the amount of subgraph sharing. In *BMDs each edge has an additional multiplicative edge weight $m \in \mathbb{Z}$, such that an edge with edge weight m leading to a node v represents a function $m \cdot f_v$. Reduction rules guarantee that functions $c_1 \cdot g$ and $c_2 \cdot g$ $(c_1, c_2 \in \mathbb{Z} \setminus \{0\})$ are represented by the same node (but by different edges).

3 Don't care assignment

In the following we present a solution to the problem to minimize a *BMD by assigning values to don't cares. We have to solve the following problem *DC*BMD*:



(a) *min_polynomial*



(b) *independent_dfs*

Figure 2: Example: BMD minimization.

Given: A *BMD B representing a function $f : \{0,\ 1\}^n \to \mathbb{Z}$ and a BDD C representing a function $c : \{0,1\}^n \to \{0,1\}$.
Find: A *BMD B' representing a function $f' : \{0,1\}^n \to \mathbb{Z}$, such that $f \cdot c = f' \cdot c$ and B' has the minimum number of nodes among all *BMDs fulfilling the same property (and respecting the same variable order).

The corresponding problem for BMDs instead of *BMDs is called *DCBMD.*

*DC*BMD* and *DCBMD* are hard problems, more precisely we can prove the following theorem:

Theorem 3.1 *DC*BMD and DCBMD are NP complete.*

Proof: The proof that *DC*BMD* and *DCBMD* are NP–hard is done by a reduction from the graph colorability problem. *DC*BMD*, *DCBMD* $\in NP$ is shown using WLCDs [18]. For details see Appendix A. □

Because of this complexity result we are looking for a *heuristic* solution of *DC*BMD* in the following.

3.1 Method *min_polynomial*

Our first method *min_polynomial* is motivated by the relationship between BMDs over variables $x_1, \ldots, x_n$ and polynomials over $x_1, \ldots, x_n$: The rule to evaluate BMDs directly implies a method to derive the polynomial representing the same function as the BMD. E.g. the function from Figure 1(a) is equal to $(1 + x_2 \cdot 5) + x_1 \cdot 4 = 1 + 5x_2 + 4x_1$. In general the polynomial contains the term $c \cdot x_1^{\epsilon_1} \cdot \ldots \cdot x_n^{\epsilon_n}$ ($x_i^1 = x_i$ and $x_i^0 = 1$) if and only if the node reached by $(\epsilon_1, \ldots, \epsilon_n)$ is terminal $c \neq 0$.

It is easy to see that the size of the BMD B representing function f_B is always less or equal to the size of the polynomial[3] representing f_B. Since *BMDs can be obtained from BMDs by reduction, this is clearly also true for *BMDs.

Our first method consists in a (heuristic) minimization of the size of this polynomial, which is an upper bound on the BMD and the *BMD size. For vectors $(\epsilon_1, \ldots, \epsilon_n)$, such that the terminal reached by $(\epsilon_1, \ldots, \epsilon_n)$ is $c \neq 0$, we try to use don't cares to change the value of the terminal to zero. If $(\epsilon_1, \ldots, \epsilon_n)$ is a don't care vector, i.e. $dc(\epsilon_1, \ldots, \epsilon_n) = 1$, we change the function value $f_B(\epsilon_1, \ldots, \epsilon_n)$ such that the terminal reached by $(\epsilon_1, \ldots, \epsilon_n)$ will be 0. Using the formula of Lemma 2.1 it is clear that we just have to set for the changed function $f_{B'}$

$$f_{B'}(\epsilon_1, \ldots, \epsilon_n) = f_B(\epsilon_1, \ldots, \epsilon_n) - c$$

to achieve this goal. After that we must not forget to adjust the values of other terminals according to this change of $f_B(\epsilon_1, \ldots, \epsilon_n)$, since the value of $f_B(\epsilon_1, \ldots, \epsilon_n)$ has an impact on all terminals, which are reached by vectors $\gamma \geq \epsilon$.

The main idea of our method *min_polynomial* is illustrated in Figure 2(a). Figure 2(a) shows a BMD for the function $f : \{0,1\}^2 \to \mathbb{Z}$ with polynomial $1 + 4x_2 + 3x_1 + 8x_1x_2$. There are two don't care vectors: $dc(0,0) = dc(1,1) = 1$.

[3]The size of a polynomial is defined as the number of constants, variable names and operators $+$ and $\cdot$ in the polynomial.

```
*BMD **function** $min\_polynomial$(*BMD $B$, BDD $DC$)
**if** $DC = \boxed{1}$ **then return** $\boxed{0}$ **fi**;
**if** $DC = \boxed{0}$ **then return** $B$ **fi**
**if** $B = constant$ **then return** $B$ **fi**
**if** computed table contains entry $result$ for $(B, DC)$ **then return** $result$ **fi**
Let $v$ be top variable of $B$ and $DC$,
$B_{low} = B|_{v=0}, B_{high} = B|_{v=1} - B|_{v=0}$,
$DC_{low} = DC|_{v=0}, DC_{high} = DC|_{v=1}$
$B'_{low} := min\_polynomial(B_{low}, DC_{low})$
$B'_{high} := min\_polynomial(B_{high} + (B_{low} - B'_{low}), DC_{high})$
$B' = B'_{low} + v \cdot B'_{high}$
**if** $size(B') \geq size(B)$ **then** $B' = B$ **fi**
insert entry $B'$ for $(B, DC)$ in computed table
**return** $B'$
```

Figure 3: Pseudo code for *min_polynomial*.

The don't care values for $(0,0)$ and $(1,1)$ are represented in the BMD by the shaded boxes of terminals 1 and 8. At first, we set terminal 1, which is reached by $(0,0)$ to 0. To achieve this we make use of the don't care vector $(0,0)$ and change $f(0,0)$ by adding -1. Then we have to propagate the change to all terminals which are reached by vectors $> (0,0)$. According to the formula of Lemma 2.1 we have to change terminal 4 by adding 1, terminal 3 by adding 1 and terminal 8 by adding -1. The resulting values for the terminals are given in Figure 2(a) in the row *1st dc* below the original terminals. Finally we make use of the don't care $(1,1)$ by adding -7 to $f(1,1)$ resulting in a 0–terminal reached by $(1,1)$. Since there is no vector greater than $(1,1)$, we do not have to propagate the change in this case and the resulting terminals are shown in the second row *2nd dc* below the original terminals. Finally, we obtain a changed function with polynomial $5x_2 + 4x_1$. The reduced version of the resulting BMD is shown on the right hand side of Figure 2(a).

The order of processing the different don't care values in the example was not arbitrary: Since we process the terminals from left to right the propagation of changes due to other don't care assignments cannot destroy the zeros we have already set. For this reason our recursive procedure processes the *BMD in a depth–first manner following low edges before high edges. Pseudo code of the resulting recursive procedure *min_polynomial* to minimize a *BMD B using don't cares specified by a BDD DC is given in Figure 3. Note that in line 10 the propagation of the changes made to B_{low} is performed by adding $B_{low} - B'_{low}$ to B_{high} before applying *min_polynomial* to B_{high}.

3.2 Method *independent_dfs*

The second method is motivated by the "matching siblings" heuristics from [20]. This heuristics was introduced to minimize BDDs in a recursive procedure. When the procedure processes a BDD node v, it tries to assign don't cares in such a way that $low(v)$ and $high(v)$ become identical. If this is possible, we have to keep this subgraph only once and additionally – because of the BDD reduction rules – node v can be removed, because the subfunction is now independent from variable $index(v)$.

Since BMDs use positive Davio decomposition instead of Shannon decomposition, the function represented by a node v cannot be made independent from variable $index(v)$ by changing $low(v)$ and $high(v)$ to make them identical. Here we try to make use of don't cares to change $high(v)$, such that it becomes 0. Then, the function represented by v is independent from $index(v)$ and we can delete $high(v)$ and (according to BMD reduction rules) also node v.

```
*BMD function independent_dfs(*BMD $B$, BDD $DC$)
if $DC = \boxed{1}$ then return $\boxed{0}$ fi;
if $DC = \boxed{0}$ then return $B$ fi
if $B = constant$ then return $B$ fi
if computed table contains entry $result$ for $(B, DC)$ then return $result$ fi
Let $v$ be top variable of $B$ and $DC$,
$B_{low} = B|_{v=0}, B_{high} = B|_{v=1} - B|_{v=0}$,
$DC_{low} = DC|_{v=0}, DC_{high} = DC|_{v=1}$
$(success, B_{low,diff}) := check\_zero(B_{high}, DC_{low}, DC_{high})$
if $success$ then
            $B' = independent\_dfs(B_{low} + B_{low,diff}, DC_{low} \cdot DC_{high})$
      else
            $B'_{low} := independent\_dfs(B_{low}, DC_{low})$
            $B'_{high} := independent\_dfs(B_{high} + (B_{low} - B'_{low}), DC_{high})$
            $B' = B'_{low} + v \cdot B'_{high}$
fi
if $size(B') \geq size(B)$ then $B' = B$ fi
insert entry $B'$ for $(B, DC)$ in computed table
return $B'$
```

Figure 4: Pseudo code for *independent_dfs*.

Thus, we have to check for a node v, which is reached by $(\epsilon_1, \ldots, \epsilon_l)$, whether the node function can be made independent from variable x_{l+1} by exploiting don't cares from $dc|_{x_1=\epsilon_1,\ldots,x_l=\epsilon_l}$. Figure 2(b) illustrates the method using the same example as in Figure 2(a). At the beginning we check whether the root node v can be made independent from x_1 by using don't cares, which is equivalent to the question, if we can set $high(v)$ to zero. To do this we can exploit don't cares both from $dc|_{x_1=0}$ and from $dc|_{x_1=1}$, i.e. both the don't cares at $(0,0)$ and $(1,1)$ in this example. The terminal reached by $(1,0)$ cannot be set to 0 using don't cares from $dc|_{x_1=1}$, but it is possible to use don't care $(0,0)$ (adding 3 to $f(0,0)$) to set this terminal to 0. Then we use don't care $(1,1)$ to set the terminal reached by $(1,1)$ to 0 and in fact, it is possible to make the root node independent from x_1. The changed values for the terminals are given in Figure 2(b) in the row below the original terminals. The reduced BMD is given on the right hand side of Figure 2(b). It is easy to see that it is not possible to make the remaining node independent from x_2, since there are no don't cares which could be exploited. (Note that also the don't care $(0,0)$ must not be used in the minimization of this node, since it was already used to make the root function independent from x_1. Exploitation of don't care $(0,0)$ could make the function depend on x_1 again.)
The check, whether a function of a node v, which is reached by $(\epsilon_1, \ldots, \epsilon_l)$, can be made independent from variable x_{l+1} using $dc|_{x_1=\epsilon_1,\ldots,x_l=\epsilon_l}$ can be formulated as a recursive procedure, which checks first if the low son can be set to 0 and then if the high son can be set to 0. This check is used in a depth–first traversal of the *BMD. Whenever we reach a node which can be made independent from its top variable, we perform the modification and the effect of the change is propagated similar to procedure *min_polynomial.*

Pseudo code for procedure *independent_dfs*, which minimizes a *BMD B using a don't care set given by DC, is shown in Figure 4. In line 9 the algorithm checks whether the high son B_{high} of a node labeled by variable v can be set to 0 or not. For this check don't cares from two sets can be used: One set is represented by $DC_{high} = DC|_{v=1}$ and the other set is represented by $DC_{low} = DC|_{v=0}$ (see also example from Figure 2(b)). The check is done by a procedure *check_zero*. *check_zero* returns a Boolean variable $success$, which indicates, whether the check was

```
1  (boolean, *BMD) function check_zero(*BMD $B_H$, BDD $DC_L$, BDD $DC_H$)
2  if $B_H = \boxed{0}$ or $DC_H = \boxed{1}$ then return $(1, \boxed{0})$ fi
3  if $B_H = constant$ and $DC_H = \boxed{0}$ and $DC_L = \boxed{1}$ then return $(1, B_H)$ fi
4  if $DC_H = \boxed{0}$ and $DC_L = \boxed{0}$ then return $(0, \boxed{0})$ fi
5  if computed table contains entry result for $(B_H, DC_L, DC_H)$ then return result fi
6  Let $v$ be top variable of $B_H$, $DC_L$ and $DC_H$,
7  $B_{H,low} = B_H|_{v=0}, B_{H,high} = B_H|_{v=1} - B_H|_{v=0}$,
8  $DC_{L,low} = DC_L|_{v=0}, DC_{L,high} = DC_L|_{v=1}, DC_{H,low} = DC_H|_{v=0}, DC_{H,high} = DC_H|_{v=1}$
9  $(success, B_{L\_diff,low}) := check\_zero(B_{H,low}, DC_{L,low}, DC_{H,low})$
10 if $success = 0$ then return $(0, \boxed{0})$ fi
11 $(success, B_{L\_diff,high}) := check\_zero(B_{H,high} + B_{H,low}, DC_{L,high}, DC_{H,high})$
12 if $success = 0$ then return $(0, \boxed{0})$ fi
13 $B_{L\_diff} = (1 - v) \cdot B_{L\_diff,low} + v \cdot B_{L\_diff,high}$
14 insert entry $(1, B_{L\_diff})$ for $(B_H, DC_L, DC_H)$ in computed table
15 return $(1, B_{L\_diff})$
```

Figure 5: Pseudo code for *check_zero*.

successful or not, and a *BMD $B_{low,diff}$. If the check is not successful ($success = 0$ in line 10) the algorithm proceeds like procedure *min_polynomial*. If the check is successful, i.e. if B_{high} can be set to 0, the exploitation of don't cares from DC_{low} has to be taken into account: Exploiting don't cares from DC_{low} means changing the negative cofactor to set B_{high} to 0. These changes are returned as a *BMD $B_{low,diff}$ by the procedure *check_zero*. Thus we have to minimize $B_{low} + B_{low,diff}$ instead of B_{low} in line 11. The don't cares, which we are allowed to use in line 11, are not given by DC_{low}, but only by $DC_{low} \cdot DC_{high}$, since we have to keep the result B' in line 11 independent from variable v.

For completeness, pseudo code for the procedure *check_zero* which checks, whether don't cares can be used to set the function of a node to 0, can be found in Figure 5.

4 Experimental results

We implemented the two methods for *BMD minimization based on `wld`, an experimental Word-Level DD package developed at University of Freiburg [11] and performed experiments to compare the different approaches. The experiments were performed using a SPARC UltraII with a memory limit of 400 MB.

To generate incompletely specified functions from completely specified functions, we used a method proposed in [6]: After collapsing each benchmark circuit to two level form, we randomly selected cubes in the on-set with a probability of 40% to be included into the don't care set. The cubes which were not selected to be included in the don't care set were used to construct a *BMD to represent a weighted sum of the output functions (output i weighted by 2^i). For the don't care set a BDD was computed. As variable order we used the initial order given in the benchmark specification. The results are summarized in Table 1. In the first column the benchmark circuit is given, in the second column the number of primary inputs and in the third column the number of primary outputs. Column 4 shows the number of BDD nodes needed to represent the don't care set and column 5 the number of nodes needed to represent the initial *BMD. Columns 6–8 give the *BMD sizes after minimization. Three different methods are compared: For comparison we give in column *az* the simple method to set all don't care input vectors to function value 0, which can be done by computing $f_B \cdot \overline{dc}$. Column *mp* gives the results for our procedure *min_polynomial* and column *dfs* the results for our procedure *independent_dfs*. Columns 9–11 give the

Circuit	#PI	#PO	$\lvert DC \rvert$	$\lvert$*BMD$\rvert$	$\lvert$*BMD$_{min}\rvert$			ratio $\frac{\lvert *\text{BMD}_{min}\rvert}{\lvert *\text{BMD}\rvert}$			Time		
					az	mp	dfs	az	mp	dfs	az	mp	dfs
5xp1	7	10	15	76	19	12	3	0.250	0.157	0.039	0:00	0:00	0:00
9symml	9	1	97	223	242	183	182	1.085	0.820	0.816	0:09	0:00	0:00
alu2	10	6	91	401	372	139	147	0.927	0.346	0.366	0:30	0:01	0:01
apex7	49	37	120	1390	2305	118	49	1.658	0.084	0.035	0:08	1:28	3:27
c8	28	18	126	346	336	17	13	0.971	0.049	0.037	0:02	1:23	0:02
mux	21	1	5798	60	47	34	34	0.783	0.566	0.566	0:00	0:06	0:18
pcler8	27	17	34	44	61	32	21	1.386	0.727	0.477	0:01	0:00	0:09
rd73	7	3	36	89	87	43	36	0.977	0.483	0.404	0:02	0:00	0:00
rd84	8	4	65	196	200	114	81	1.020	0.581	0.413	0:15	0:00	0:00
sao2	10	4	52	128	96	47	37	0.750	0.367	0.289	0:01	0:00	0:00
z4ml	7	4	30	69	87	30	26	1.260	0.434	0.376	0:00	0:00	0:00
			$\sum$	3022	3852	769	629	1.247	0.254	0.208			

Table 1: Results for don't care minimization.

ratios "size of minimized *BMD divided by size of initial *BMD", again for the three different methods. Finally the corresponding CPU times are given in columns 12–14 in format minutes:seconds, rounded to seconds.

The results show that setting all don't cares to 0 (columns *az*) is not a successful method. On the average the sizes even increase by 24.7%. In contrast, our two methods for don't care minimization are both very effective in minimizing the *BMD sizes: Method *min_polynomial* (columns *mp*) is able to reduce *BMD sizes by 74.6% on the average and method *independent_dfs* (columns *dfs*) reduces the sizes even by 79.2%. Columns 13 and 14 show that these results can be achieved within a small amount of run time.

5 Conclusions and future work

We presented two heuristic methods for don't care minimization of *BMDs. Experimental results proved them to be very effective in reducing *BMD sizes within a small amount of CPU time.

At the moment we are working on a modified version of method *independent_dfs*, which is based on the observation that in contrast to BDDs [20] for *BMDs the order in which we process the nodes can influence the quality of the result due to the propagation of the change. Setting the high son of a node v to 0 can destroy the possibility to set the high son of another node v' to 0. Since the subgraph of the high son of a node at a higher level in the *BMD will be larger on the average, we expect that the gain of setting the high son of such a node to 0 is also larger. Therefore nodes at higher levels should be processed first leading to a breadth-first traversal of the *BMD instead of a depth-first traversal.

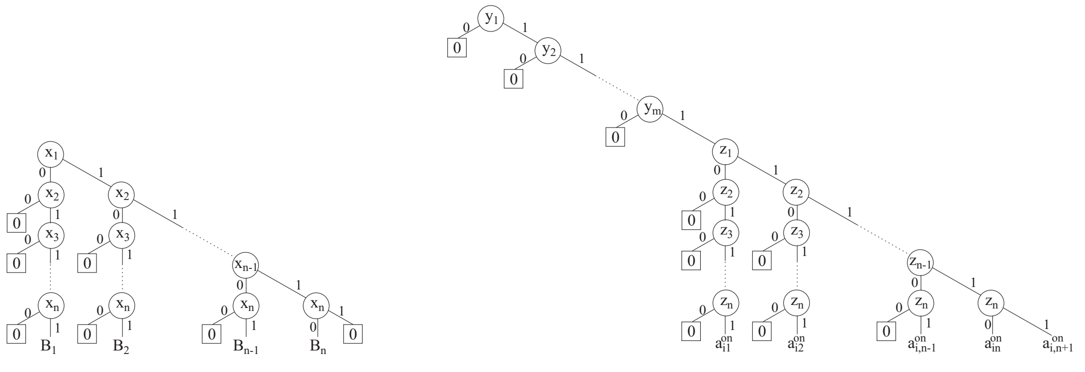
Moreover, we are working on an application of our *BMD minimization in the verification of Pentium style integer dividers to keep peak memory consumption small during backward construction [10]. Don't cares are computed by an iterative image computation for the different add&shift stages.

A Proof of Theorem 3.1

We prove Theorem 3.1 for the decision problem versions *DCBMD'* and *DC*BMD'* of *DCBMD* and *DC*BMD*.

DCBMD': Given a BMD B representing a function $f : \{0,1\}^n \to \mathbb{Z}$, a BDD C representing a function $c : \{0,1\}^n \to \{0,1\}$ and a constant $s \in I\!N$. Is there a BMD B' of size $\leq s$ (with the same variable order) representing a function $f' : \{0,1\}^n \to \mathbb{Z}$, such that $f \cdot c = f' \cdot c$?

*DC*BMD'*: Given a *BMD B representing a function $f : \{0,1\}^n \to \mathbb{Z}$, a BDD C representing a function $c : \{0,1\}^n \to \{0,1\}$ and a constant $s \in I\!N$. Is there a *BMD B' of size $\leq s$ (with the same variable order) representing a function $f' : \{0,1\}^n \to \mathbb{Z}$, such that $f \cdot c = f' \cdot c$?



(a) Top part of B (b) Structure of sub–BMDs B_i

Figure 6: Definition of BMD B.

At first, we prove the theorem for problem *DCBMD*$'$.

Proof: The first part is to prove that *DCBMD*$'$ is NP-hard. The proof uses ideas and proof techniques from [17] and [2]. In [17] Sauerhoff and Wegener prove that minimizing the BDD size of incompletely specified functions is NP-hard and in [2] Bollig, Löbbing, Sauerhoff and Wegener prove that the same problem is NP-hard for FDDs. Since FDDs also use the positive Davio decomposition, the proof can be adapted to BMDs.

Similar to [2] and [17] we construct a reduction from the well-known graph colorability problem (GC) to *DCBMD*$'$.

An instance of GC is a connected undirected graph $G = (V, E)$ with the property that E does not contain any edges $\{v, v\}$, $v \in V$, and a number k. The problem is to decide whether G has a k-coloring, i.e. whether there is a function $\phi : V \to \{1, \ldots, k\}$, such that the endpoints of the edges are colored differently ($\phi(v) \neq \phi(w)$ for all $\{v, w\} \in E$.

Let $G = (V = \{v_1, \ldots, v_n\}, E)$ and k be the given instance for GC. The corresponding instance for *DCBMD*$'$ consists of a BMD B, a BDD C and a size bound s.

For the BMD and the BDD we use the following variables (where m is a parameter defined later on in the proof); the variables are to be tested in the given order:

$$x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_n.$$

We first describe the BMD B. This BMD realizes almost the same function as in the proof of [2]. The function values are only 0 and 1, but we have to take into account that we have to construct a BMD, i.e. a word–level data structure. We use a substructure depending on the x–variables at the top of B as a switch to choose exactly one of the sub–BMDs $B_1, \ldots, B_n$ of B (see Figure 6(a)). These sub–BMDs will correspond to the vertices of the graph G. Let f be the function computed by B and let f_i be the function computed by the sub–BMD B_i.

All the sub–BMDs B_i for $i = 1, \ldots, n$ have the structure shown in Figure 6(b). The number m of y–nodes will be needed to adjust the graph size of the sub–BMDs B_i. The part containing the z–variables again is a switch. This time, one of the constants a_{ij}^{on} ($j \in \{1, \ldots, n\}$) describing the neighborhood of vertex v_i in G is chosen by the switch. We define for $j \in \{1, \ldots, n\}$

$$a_{ij}^{on} := \begin{cases} 1, \text{ if } \{v_i, v_j\} \in E; \\ 0, \text{ otherwise.} \end{cases}$$

Note that especially $a_{ii}^{on} = 0$ for $1 \leq i \leq n$.

The value of $a_{i,n+1}^{on}$ is set to $(-1) \cdot \sum_{j=1}^{n} a_{ij}^{on}$, such that the sum of all values a_{ij}^{on} equals zero.

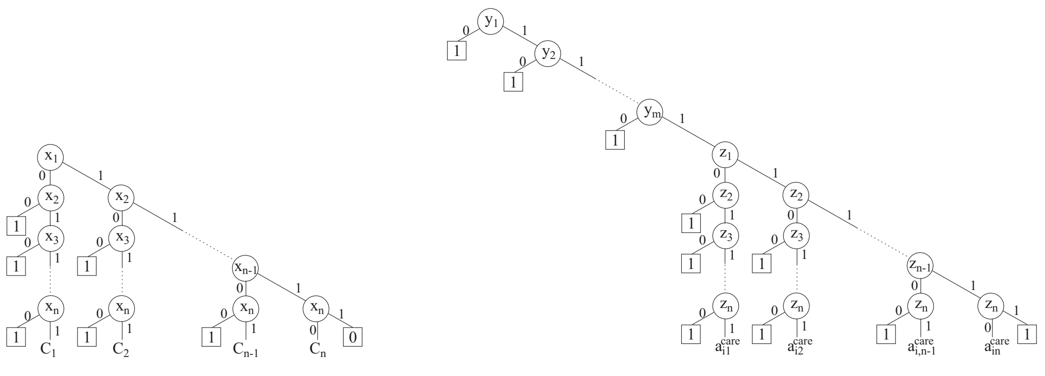

(a) Top part of C (b) Structure of sub–BDDs C_i

Figure 7: Definition of BDD C.

We have to find out now which function is represented by the thus constructed BMD. Let b_i^r denote an input vector of length r, which has a zero at the i–th position and ones everywhere else. Examining the construction of B, we find out that

$$f|_{x=\alpha} := \begin{cases} f_i & \text{, if } \alpha = b_i^n, \\ \sum_{i=1}^n f_i, & \text{if } \alpha = (1,\ldots,1), \\ 0 & \text{, if } \alpha \in \{0,1\}^n \setminus \{b_1^n,\ldots,b_n^n,(1,\ldots,1)\}. \end{cases}$$

A sub–BMD B_i obviously computes zero, if at least one of the y–variables is zero. The part containing the z–variables has the same structure as the top part of B, so it is easy to see that for $1 \leq i \leq n$, $\beta \in \{0,1\}^m$, $\gamma \in \{0,1\}^n$

$$f_i(\beta,\gamma) := \begin{cases} a_{i,j}^{on}, \text{ if } \beta = (1,\ldots,1) \text{ and } \gamma = b_j^n, \\ 0 \text{ , otherwise.} \end{cases}$$

Note that for $\beta = (1,\ldots,1)$ and $\gamma = (1,\ldots,1)$ the definition $a_{i,n+1}^{on} = (-1) \cdot \sum_{j=1}^n a_{ij}^{on}$ guarantees that $f_i(\beta,\gamma) = 0$.

The care set for function f has to be specified by a BDD C. Before we construct a BDD C to represent a characteristic function c of the care set, we define values a_{ij}^{care} as follows $(1 \leq i,j \leq n)$:

$$a_{ij}^{care} := \begin{cases} 1, \text{ if } j = i \vee \{v_i,v_j\} \in E; \\ 0, \text{ otherwise.} \end{cases}$$

The underlying graph of the BDD is similar to the BMD. The top part of the graph is given by Figure 7(a). It differs from the graph in Figure 6(a) from the fact, that terminals 0 are replaced by terminals 1 (except the last 0), for $1 \leq i \leq n$ B_i is replaced by C_i.

The graphs for the sub–BDDs C_i $(1 \leq i \leq n)$ are given by Figure 7(b). Again, the difference to the graph in Figure 6(b) lies in the fact, that terminals 0 are replaced by terminals 1, a_{ij}^{on} are replaced by a_{ij}^{care} $(1 \leq j \leq n)$ and $a_{i,n+1}^{on}$ is replaced by 1.

Let c be the function computed by C, then it is easy to see that for $\alpha,\gamma \in \{0,1\}^n$ and for $\beta \in \{0,1\}^m$

$$c(\alpha,\beta,\gamma) := \begin{cases} a_{ij}^{care}, \text{ if } \alpha = b_i^n, \beta = (1,\ldots,1), \gamma = b_j^n, (i,j \in \{1,\ldots,n\}), \\ 0 \text{ , if } \alpha = (1,\ldots,1), \\ 1 \text{ , otherwise.} \end{cases}$$

Like for f_i and B_i we use in the following the notion c_i for the function represented by C_i; $c_i = c_{x=b_i^n}$.

Finally, we choose $s := k(m + n(n+1)/2) + n(n+1)/2 + 2$ for the BMD size in the instance of *DCBMD'*. Both constructed graphs B and C have size $O(nm + n^3)$. We will fix m to $n(n+1)^2/2 + 2$ below, so that these sizes are polynomial in n. Note that strictly speaking B and C are not a BMD and a BDD, since it is possible that reduction rules are applicable to B and C to reduce the graph size. However the reduction of B to a BMD and of C to a BDD can easily be done in polynomial time and it can only reduce the size. All in all we can say that the size of the constructed instance of *DCBMD'* is polynomial in n and can be computed in polynomial time.

We have to show that

$$(G,k) \in GC \iff (B,C,s) \in DCBMD'.$$

$\Longrightarrow$: Let a k–coloring of G be given. We have to construct a BMD B' which computes a function f' such that $f \cdot c = f' \cdot c$ and whose size is bounded by s. For B' we use the same graph structure as for B.

Let f' be the function represented by the BMD B' and let f'_i be the function of the subgraph B'_i defined in the same way as in the construction of B.

For $1 \le i,j \le n$ we replace a_{ij}^{on} by

$$a'_{ij} := \begin{cases} 0, \text{ if } v_i \text{ and } v_j \text{ have the same color;} \\ 1, \text{ otherwise.} \end{cases}$$

The value of $a'_{i,n+1}$ is set to $(-1) \cdot \sum_{j=1}^n a'_{i,j}$, such that also in B' the sum of all values a'_{ij} equals zero.

We first verify that indeed $f \cdot c = f' \cdot c$. Let $\alpha = b_i^n$, $\beta = (1,\ldots,1)$ and $\gamma = b_j^n$, i.e. an input where $c(\alpha,\beta,\gamma) = a_{ij}^{care}$ and $f'(\alpha,\beta,\gamma) = a'_{ij}$. If $a_{ij}^{care} = 0$ then $(f \cdot c)(\alpha,\beta,\gamma) = (f' \cdot c)(\alpha,\beta,\gamma) = 0$. If $a_{ij}^{care} = 1$, then $j = i$ or $\{v_i,v_j\} \in E$. If $j = i$, then $f(\alpha,\beta,\gamma) = a_{ii}^{on} = 0$ and $f'(\alpha,\beta,\gamma) = a'_{ii} = 0$. If $\{v_i,v_j\} \in E$, $f(\alpha,\beta,\gamma) = a_{ij}^{on} = 1$ and $f'(\alpha,\beta,\gamma) = a'_{ij} = 1$, since in this case, v_i and v_j must have different colors.
If $\alpha = (1,\ldots,1)$, then $c(\alpha,\beta,\gamma) = 0$.
For all other choices of α, β and γ, we get $f'(\alpha,\beta,\gamma) = f(\alpha,\beta,\gamma) = 0$.

Now we will show that after applying the BMD reduction rules B' will have at most s nodes. It is easy to see that $f'_i = f'_j$, if the vertices v_i and v_j belong to the same color class. Thus all functions f'_i belonging to vertices in the same color class can be represented by the same subgraph of B'. Thus the subgraphs $B'_1 \ldots B'_n$ can be merged to at most k different subgraphs. The resulting reduced BMD has at most $k(m + n(n+1)/2) + n(n+1)/2 + 2 = s$ nodes.

$\Longleftarrow$: Now let a BMD B' for f' with $f \cdot c = f' \cdot c$ be given for which $|B'| \le s$. Let $f'_i := f'|_{x=b_i^n}$. We define a coloring of G as follows.

Two vertices v_i, v_j from G obtain the same color, iff $f'_i = f'_j$. It is easy to verify that this is a legal coloring. We have to show that from $f'_i = f'_j$ it follows that $\{v_i,v_j\} \notin E$. First, let us consider the case that $c_j(b,b_i^n) = a_{ji}^{care} = 1$, where $b = (1,\ldots,1)$. Then $f'_j(b,b_i^n) = f_j(b,b_i^n) = a_{ji}^{on}$ and, since $c_i(b,b_i^n) = a_{ii}^{care} = 1$, $f'_i(b,b_i^n) = f_i(b,b_i^n) = a_{ii}^{on} = 0$. Thus $f'_i = f'_j$ implies $a_{ji}^{on} = f'_j(b,b_i^n) = f'_i(b,b_i^n) = 0$ and $\{v_i,v_j\} \notin E$. If $c_j(b,b_i^n) = a_{ji}^{care} = 0$, $\{v_i,v_j\} \notin E$ follows directly from the definition of a_{ji}^{care}.

Now we have to show that our coloring does not use too many colors (at most k). We claim that $|B'| \ge d(m+1)$, if there are at least d pairwise different functions f'_i with $i \in \{1,\ldots,n\}$. For the proof of this claim we consider cofactors $h_{ir} := f'_i|_{y_1=1,\ldots,y_r=1}$ of f', where $i \in \{1,\ldots,n\}$ and $r \in \{0,\ldots,m\}$ (we let $h_{i0} := f'_i$). We show that these cofactors are represented in B' and that enough cofactors are pairwise different, such that they are represented by different nodes.

First we show that the cofactor functions h_{ir} are represented by nodes of B'. We consider the node, which is reached by the path (b_i^n, b)[4] with $b = (1,\ldots,1) \in \{0,1\}^r$, and show that this node represents h_{ir}. According to Lemma 2.1 the node reached by (b_i^n, b) represents the function

$$\sum_{(\alpha,\beta) \le (b_i^n, b)} (-1)^{\sum_{j=1}^n ((b_i^n)_j - \alpha_j) + \sum_{j=1}^r (b_j - \beta_j)} f'_{x_1=\alpha_1,\ldots,x_n=\alpha_n, y_1=\beta_1,\ldots,y_r=\beta_r}$$

[4]As defined in Section 2.

Since $c|_{x_1=\alpha_1,\ldots,x_n=\alpha_n} \equiv 1$, $f|_{x_1=\alpha_1,\ldots,x_n=\alpha_n} \equiv 0$ for $\alpha < b_i^n$ and also
$c_i|_{y_1=1,\ldots,y_{j-1}=1,y_j=0} \equiv 1$, $f_i|_{y_1=1,\ldots,y_{j-1}=1,y_j=0} \equiv 0$, we have
$f'|_{x_1=\alpha_1,\ldots,x_n=\alpha_n} \equiv 0$ for $\alpha < b_i^n$ and $f_i'|_{y_1=1,\ldots,y_{j-1}=1,y_j=0} \equiv 0$.

Therefore

$$\sum_{(\alpha,\beta)<(b_i^n,b)} (-1)^{\sum_{j=1}^n((b_i^n)_j-\alpha_j)+\sum_{j=1}^r(b_j-\beta_j)} f'_{x_1=\alpha_1,\ldots,x_n=\alpha_n,y_1=\beta_1,\ldots,y_r=\beta_r} \equiv 0$$

and the node reached by (b_i^n, b) represents exactly h_{ir}.

Now we consider a subset $I \subset \{1,\ldots,n\}$ of indices with $|I| = d$, such that for all $i \neq j \in I$ $f_i' \neq f_j'$. We show that for all $i \in I$, $r \in \{0,\ldots,m\}$ the cofactors h_{ir} are different. For $r_1 \neq r_2 \in \{0,\ldots,m\}$ cofactors h_{ir_1} and h_{jr_2} are different, since for $r \in \{0,\ldots,m-1\}$ h_{ir} depends on y_{r+1}: $h_{ir}|_{y_{r+1}=0} = f_i'|_{y_1=1,\ldots,y_r=1,y_{r+1}=0} \equiv 0$ as already shown above and $h_{ir}|_{y_{r+1}=1} \not\equiv 0$, since $h_{ir}|_{y_{r+1}=1,\ldots,y_m=1} = f_i'|_{y_1=1,\ldots,y_m=1}$ is not constant 0 (this follows from the fact that there is at least one outgoing edge $\{v_i, v_j\}$ of node v_i in G and therefore $f_i'|_{y_1=1,\ldots,y_m=1}(b_j^n) = a_{ij}^{on} = 1$).

Next we show that for $i \neq j \in I$ the functions h_{ir} and h_{jr} are different. For $r = 0$ $h_{i0} = f_i' \neq f_j' = h_{j0}$. Since $h_{ir}|_{y_{r+1}=0} = h_{jr}|_{y_{r+1}=0} \equiv 0$ (as shown above) $h_{ir} \neq h_{jr}$ implies $h_{i,r+1} \neq h_{j,r+1}$ and $h_{ir} \neq h_{jr}$ for all $r \in \{0,\ldots,m\}$ follows by induction.

We have defined $|I|\cdot(m+1) = d\cdot(m+1)$ pairwise different functions, which are not constant 0 and which have to be represented by nodes of B'. Since no node can represent two different functions, we have $|B'| \geq d(m+1)$.

We are now able to complete the proof for "$\Longleftarrow$". Let d be the number of equivalence classes of equal f_i' and thus the number of colors of our coloring. We know that $|B'| \geq d(m+1)$ and that $|B'| \leq s = k(m + n(n+1)/2) + n(n+1)/2 + 2$. Together with the fact that $k \leq n$, we obtain

$$d < k + \frac{(k+1)n(n+1)/2+2}{m+1} \leq k + \frac{n(n+1)^2/2+2}{m+1}$$

and setting $m := n(n+1)^2/2 + 2$ finally $d \leq k$, i.e. we use at most k colors.

It remains to prove that *DCBMD'* is in *NP*. It is possible to guess a BMD B' of size s. We have to prove that for the function f' realized by B' the check $f \cdot c = f' \cdot c$ can be done in polynomial time. To prove this, we use WLCDs [18]. According to [18] B, c and B' can be translated into WLCDs in linear time. Then we have to check whether $(f - f') \cdot c \equiv 0$. Subtraction of two WLCDs B and B' can be done in linear time and multiplication in quadratic time. The better worst case complexity of operations for WLCDs has to be paid by a more complicated equivalence check, but the check is still polynomial. Although WLCDs are not a canonical data structure, the reduction of a WLCD to a WLCD representing the same function with a minimal number of nodes can be done in polynomial time by Gaussian eliminations which are performed level by level. Since there is only one WLCD representing the 0–function, namely the empty WLCD containing no nodes at all, we simply have to check, whether the reduced WLCD has zero nodes or not. This proves that the check $f \cdot c = f' \cdot c$ can be done in polynomial time.

□

The proof for problem *DC*BMD'* can be done in a similar way by having a close look at the proof for *DCBMD'*.

Proof: (Sketch)
We construct the same functions f and c as in the proof for BMDs. To obtain a *BMD from the constructed graph B we just have to apply additional reduction rules, which can further reduce the graph size compared to BMDs, but it is clear that the construction can also be performed in polynomial time.

In the "$\Longrightarrow$"–part of the proof for *DCBMD'* we constructed from a k–coloring a BMD B of size $\leq s$. Here we construct the same graph and the additional *BMD reduction rules can make the graph only smaller.

In the "$\Longleftarrow$"–part we constructed a coloring with $d \leq k$ colors from a BMD with at most s nodes. Due to additional *BMD reduction rules this construction has to be changed slightly for *BMDs. Like in the proof for BMDs we consider functions h_{ir}, $i \in \{1,\ldots,n\}$, $r \in \{0,\ldots,m\}$. Functions h_{ir} are represented by nodes in the *BMD.[5] Now two vertices

[5]Here we use for functions $g \not\equiv 0$ the notion "g is represented by *BMD node v" iff $g = k \cdot f_v$ for $k \in \mathbb{Z} \setminus \{0\}$. Note that in a *BMD there cannot be two nodes which represent the same function in this sense.

v_i, v_j from G obtain the same color, iff h_{i0} and h_{j0} are represented by the same *BMD node. As in the BMD proof we have to prove that this is a legal coloring. We have to show that from the fact that h_{i0} and h_{j0} are represented by the same *BMD node, i.e. $\frac{1}{k_i} \cdot h_{i0} = \frac{1}{k_j} \cdot h_{j0} \Leftrightarrow k_j \cdot h_{i0} = k_i \cdot h_{j0}$ for $k_i, k_j \in \mathbb{Z} \setminus \{0\}$, it follows that $\{v_i, v_j\} \notin E$. Again the first case is $c_j(b, b_i^n) = a_{ji}^{care} = 1$. Then $h_{j0}(b, b_i^n) = f_j(b, b_i^n) = a_{ji}^{on}$ and, since $c_i(b, b_i^n) = a_{ii}^{care} = 1$, $h_{i0}(b, b_i^n) = f_i(b, b_i^n) = a_{ii}^{on} = 0$. Thus $k_j \cdot h_{i0} = k_i \cdot h_{j0}$ implies $k_i \cdot a_{ji}^{on} = k_i \cdot h_{j0}(b, b_i^n) = k_j \cdot h_{i0}(b, b_i^n) = k_j \cdot 0 = 0$. Thus $a_{ji}^{on} = 0$, since $k_i \neq 0$ and therefore $\{v_i, v_j\} \notin E$. If $c_j(b, b_i^n) = a_{ji}^{care} = 0$, $\{v_i, v_j\} \notin E$ follows from definition.

To prove that $d \leq k$ we have to prove for the *BMD B' that $|B'| \geq d(m+1)$. Again, this is proven by the fact, that in the *BMD $d(m+1)$ different functions $h_{ir} \not\equiv 0$ are represented by different nodes. As in the case of BMDs we conclude that functions h_{ir} and $h_{jr'}$ with $r \neq r'$ are represented by different nodes, since the functions essentially depend on a different set of variables. Again we consider a subset $I \subset \{1, \ldots, n\}$ of indices with $|I| = d$, such that for all $i \neq j \in I$ h_{i0} and h_{j0} are represented by different nodes, i.e. there exist no $k_i, k_j \in \mathbb{Z} \setminus \{0\}$ with $k_j \cdot h_{i0} = k_i \cdot h_{j0}$. We show that for $i \neq j \in I$ the functions h_{ir} and h_{jr} $(r \in \{0, \ldots, m\})$ are represented by different nodes. Since $h_{ir}|_{y_{r+1}=0} = h_{jr}|_{y_{r+1}=0} \equiv 0$, $\exists k_i, k_j \in \mathbb{Z} \setminus \{0\}$ with $k_j \cdot h_{i,r+1} = k_i \cdot h_{j,r+1}$ would imply $k_j \cdot h_{ir} = k_i \cdot h_{jr}$ and by induction $k_j \cdot h_{i0} = k_i \cdot h_{j0}$, which is a contradiction. From $|B'| \geq d(m+1)$ we conclude again $d \leq k$.

The proof, that *DC*BMD'* is in *NP*, can be done in a completely analogeous manner: Also *BMDs can be transformed to WLCDs in linear time and the remaining arguments are the same.

$\square$

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